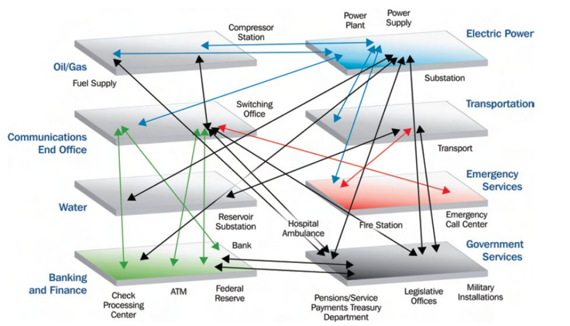
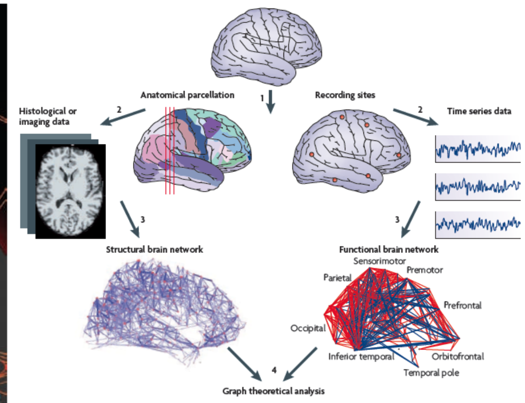



FIGURE 3.1 Connections and interdependencies across the economy. Schematic showing the interconnected infrastructures and their qualitative dependencies and interdependencies. SOURCE: Department of Homeland Security, National Infrastructure Protection Plan, available at http://www.dhs.gov/xprevprot/programs/editorial_0827.shtm.




*Cambridge Networks Day*

*Cambridge, 7 May 2013*

# Statistical Mechanics of Multiplex Networks

Ginestra Bianconi

*School of Mathematical Sciences, Queen Mary University of London, London, UK*

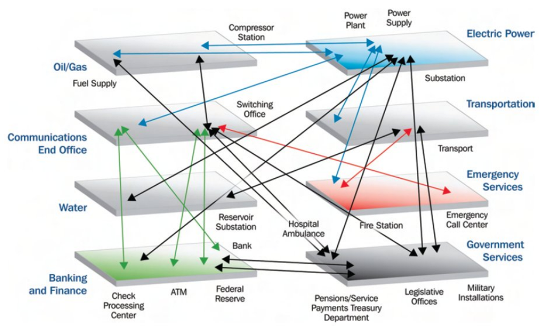
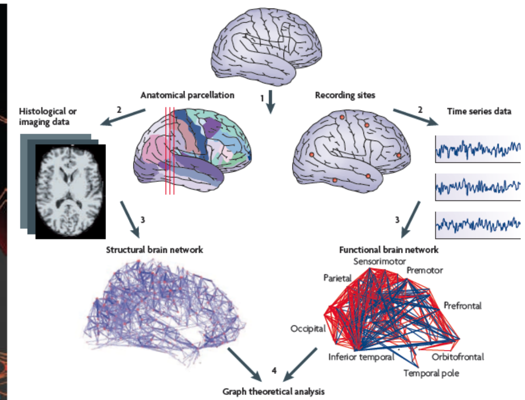



FIGURE 3.1 Connections and interdependencies across the economy. Schematic showing the interconnected infrastructures and their qualitative dependencies and interdependencies. SOURCE: Department of Homeland Security, National Infrastructure Protection Plan, available at http://www.dhs.gov/xprevprot/programs/editorial_0827.shtm.




The function of many complex

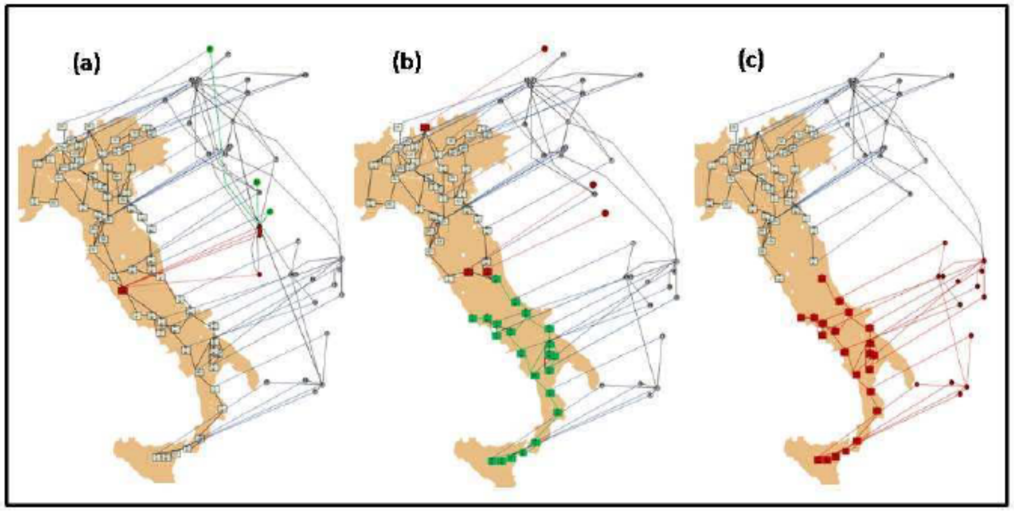
**technological social and biological** systems

depends on the non-trivial interactions between

interacting networks

# Interacting infrastructure networks

*Complex infrastructures are interdependent and a failure in one network can generate a cascade of failures in the Interdependent networks*




**Buldyrev et al. Nature 2010**

# Interacting Transportation networks

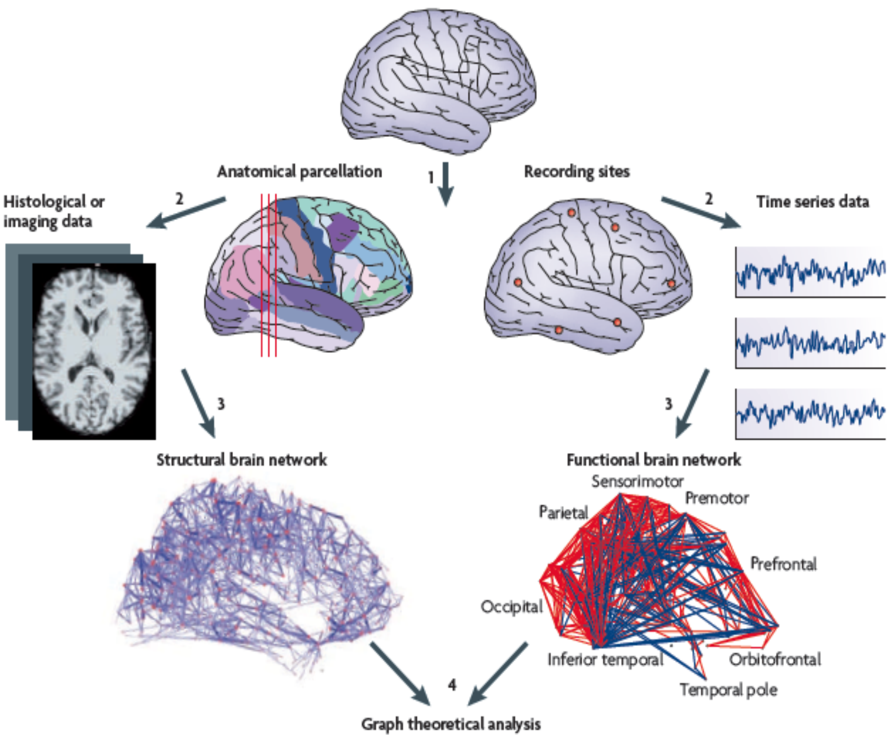
*Transportation networks are another major example of interacting networks.*

*Here*
*blue lines represent short-range commuting flow by car or train*
*the red lines indicate airline flow for few selected cities*

B. GONÇALVES ET AL., INDIANA UNIV.

**Vespignani Nature 2010**

# Interacting and multiplex Brain networks




The brain function is determined

at the same time

by

the structural brain network

and

the functional brain network,

*Bullmore Sporns 2009*

# Interacting
# Social networks

*Social networks are interacting and overlapping with profound implications for community detection algorithms*

Y.Y. Ahn et al. Nature 2010

# Interacting and multiplex networks

In order to

**model, predict and control**

**complex networks**

we need  to understand

the effect of interdependencies between networks

and

we need to fully characterize

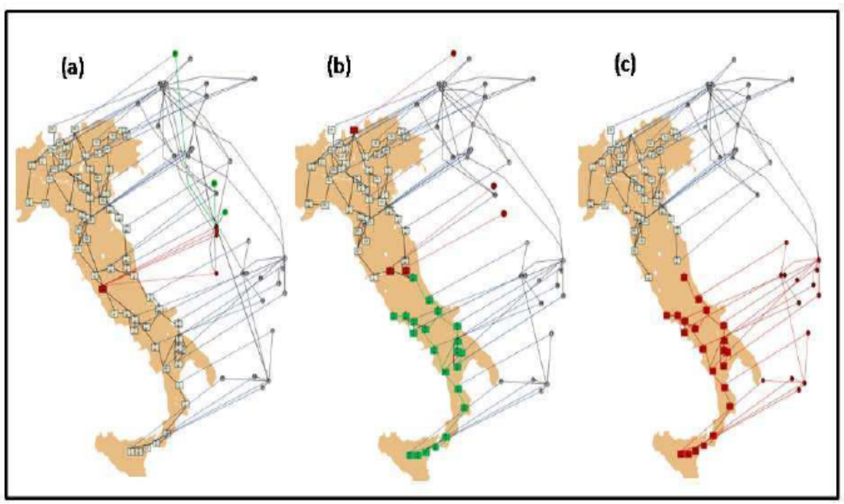
the evolution and dynamics of

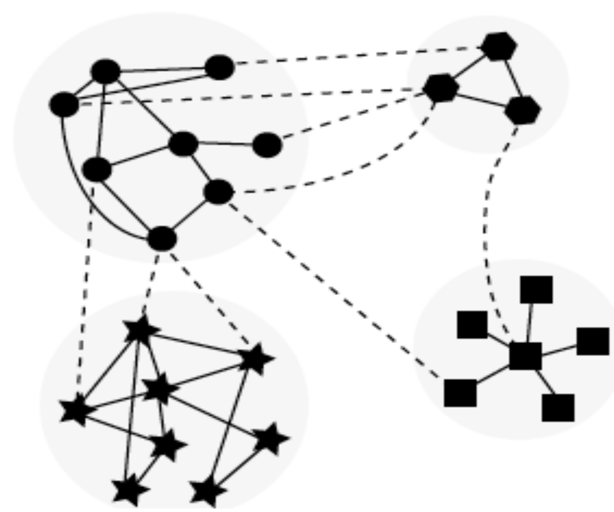
the

**networks of networks**

# Interacting networks

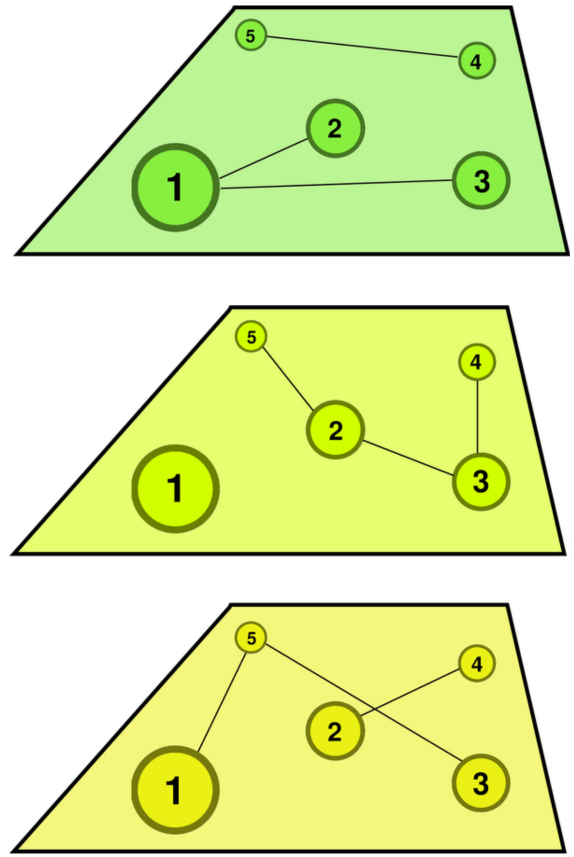
- Two or more interacting networks are formed by different nodes (ex. Power-grid network and Internet)
- but there might be complex interactions and interdependencies between the nodes



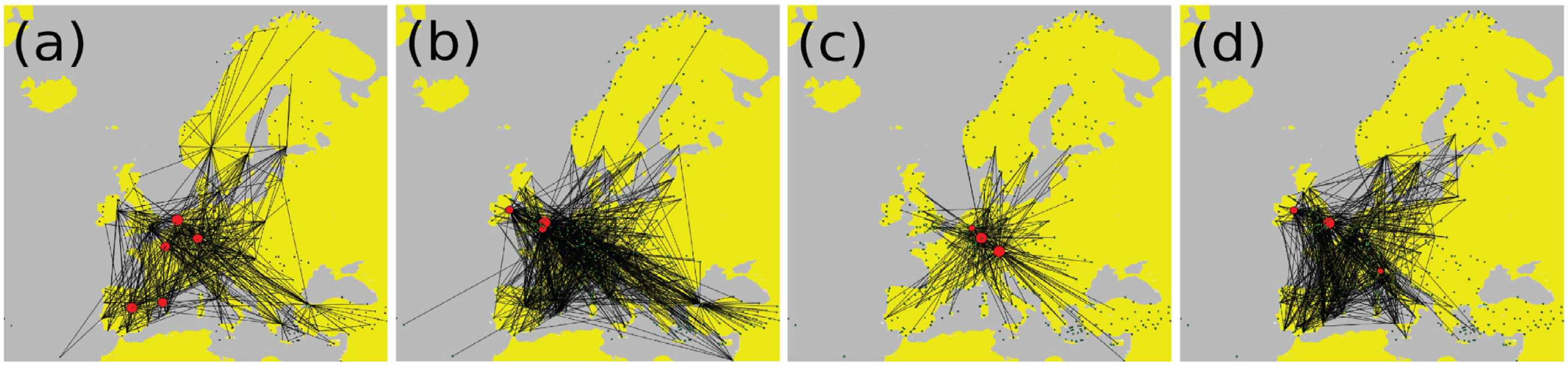



# Multiplex

- A multiplex is formed by a set of nodes that are present at the same time on different networks,
- A multiplex is formed by M layers (in the figure M=3)
- Each layer is formed by a network



# The airport network is a multiplex

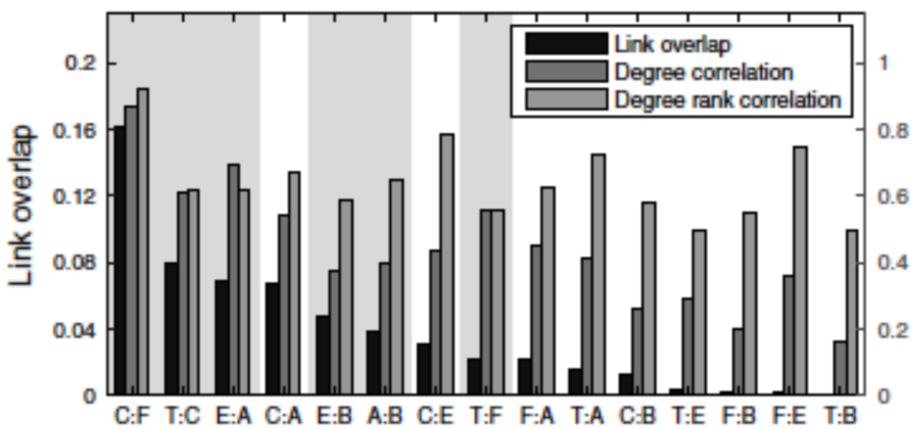


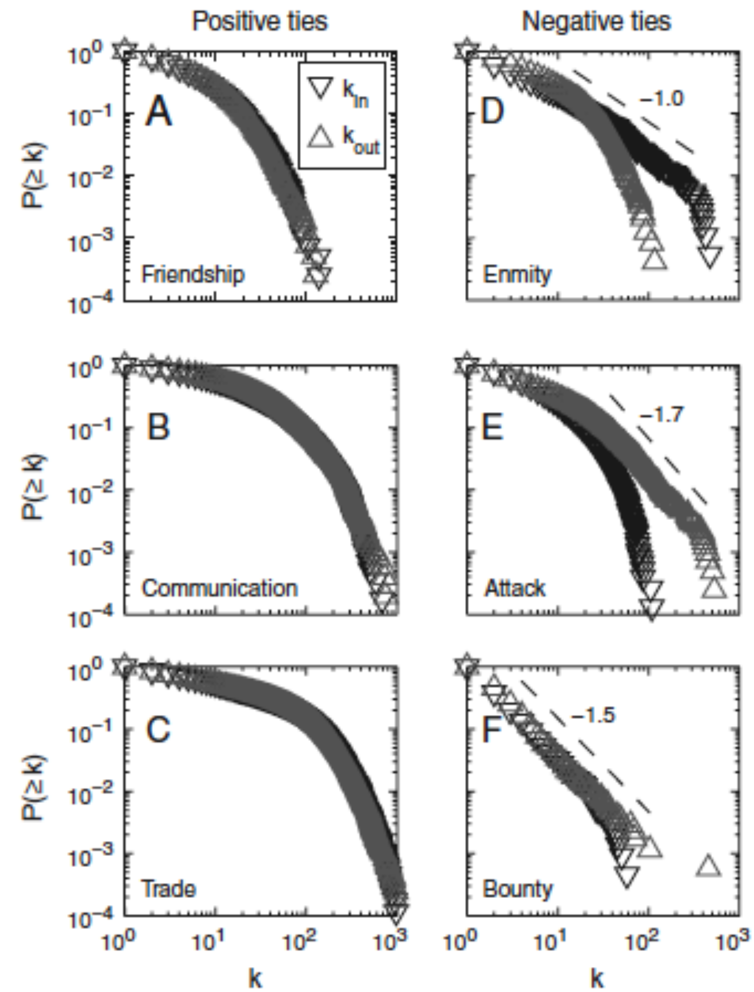

- (a) Only links belonging to all airline companies are plotted
- (b) The combined network where only nodes of degree k>75 have been plotted
- (c) A major airline network
- (d) Low cost airline network

**Cardillo et al. Scientific Reports (2013).**

# The in silico multiplex social social network of an online game

- In this online game agents can belong to different networks Friendship, Communication, Trade, Enmity, Attack and Bounty networks




Szell et al. PNAS 2010

# Representation of a multiplex

The straightforward representation a multiplex of N nodes formed by M layers is

by means of the set of M adjacency matrices

$$\boxed{a^{\alpha}}$$

with $\alpha$=1, 2, … M and matrix elements

$$\boxed{a_{ij}^{\alpha} = \begin{cases} 1 & \text{if node } i \text{ and node } j \text{ are linked in layer } \alpha \\ 0 & \text{otherwise} \end{cases}}$$

# Multiplex Models

*1) growing multiplex model*

*2) ensembles of multiplex*

# Class of network models

- **Growing networks:**
  - **Preferential attachment**

    *Barabasi & Albert 1999,*

    *Dorogovtsev & Mendes 2000,*

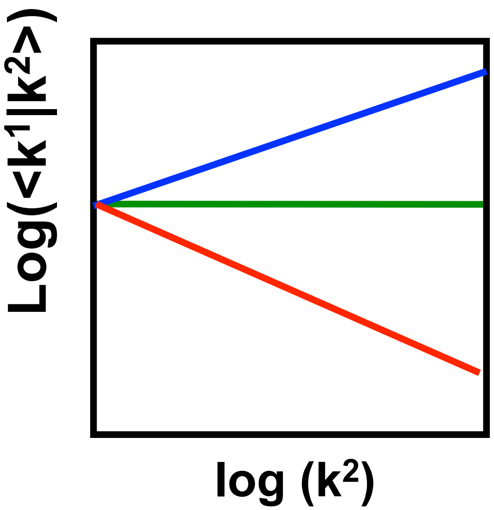
    *Bianconi & Barabasi 2001*

- Static networks:
  - **Hidden variables mechanism**

    *Bollobas 1979, Chung & Lu 2002,*

    *Caldarelli et al. 2002, Park & Newman 2003*

# Conditional average degree of a node in one layer (case of a duplex, i.e. two layers)




Positive degree correlations

No degree correlations

Negative degree correlations

$$\langle k^1 | k^2 \rangle = \sum_{k^1} k^1 P(k^1, k^2)$$

**$k^1$ degree in network 1, $k^2$ degree in network 2**
**$P(k^1,k^2)$ probability that a node has degree $k^1$ in one layer and $k^2$ in the other layer**

# Growing multiplex (duplex)

- GROWTH

At each time a new node is added to the multiplex.

Every new node has a copy in each layer and has m links in each layer.

- LINEAR PREFERENTIAL ATTACHMENT

The probability that the new link is added to node i in layer $\alpha$

is given by $\Pi^{\alpha}$ with

$$\Pi_i^1 \propto a k_i^1 + (1-a) k_i^2$$

$$\Pi_i^2 \propto (1-b) k_i^1 + b k_i^2$$

and a,b $\leq$ 1.

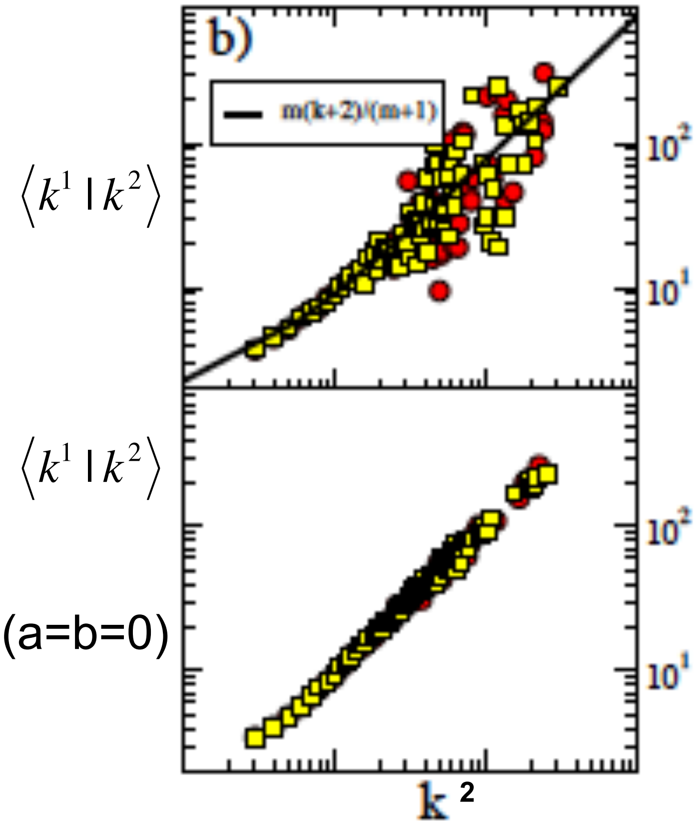
# Degree correlations

**Nicosia et al arxiv:1302.7126**

- Case a=b=1  Exact solution

$$P(k^1,k^2)=\frac{2\Gamma(2+2m)\Gamma(k^1)\Gamma(k^2)\Gamma(k^1+k^2-2m+1)}{\Gamma(m)\Gamma(m)\Gamma(k^1-m+1)\Gamma(k^2-m+1)}$$

$$\left\langle k^1 \mid k^2\right\rangle=\frac{m}{1+m}(k^2+2)$$




- For general a,b solving in the mean-field approximation it can be obtained

$$\left\langle k^1 \mid k^2\right\rangle \propto k^2$$

- From the simulation results it is possible to conclude that the degree correlations are minimal in the a=b=1 case

# Growing multiplex (duplex)

- GROWTH

At each time a new node is added to the multiplex. Every new node has a copy in each layer and has m links in each layer.

- SEMILINEAR PREFERENTIAL ATTACHMENT

The probability that the new link is added to node i in layer $\alpha$ is given by $\Pi^{\alpha}$ with

$$\Pi_i^1 \propto a k_i^1 + (1-a)$$
$$\Pi_i^2 \propto (1-b) k_i^1 + b$$

and a,b $\leq$ 1.

# Degree correlations

- Case a=b=1 Exact solution

**Nicosia et al arxiv:1302.7126**

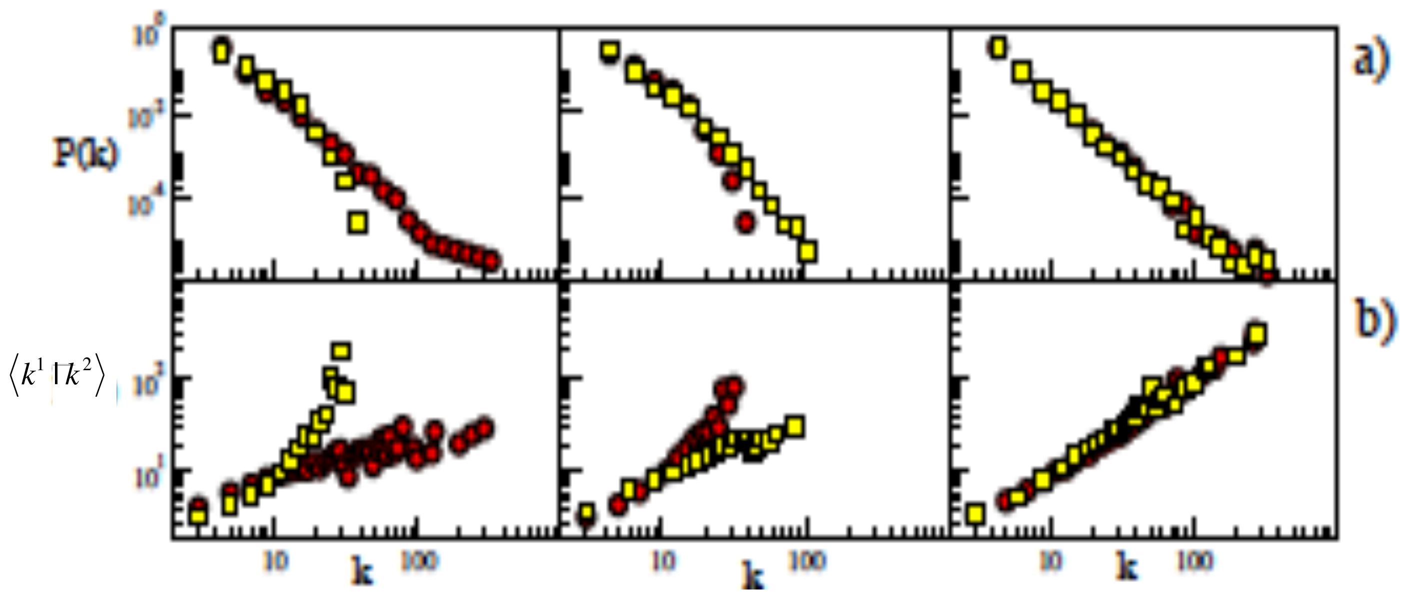
$$P(k^1,k^2)=\frac{\Gamma(k^1)}{\Gamma(m+1)\Gamma(k^1-m+1)}\sum_{n=0}^{k-m}\binom{k-m}{n}\left(\frac{2m}{1+2m+k^1-n}\right)^{k^2-m}(-1)^{k^1-m-n}$$

$$\langle k^1 \mid k^2\rangle = m\left(\frac{2(m+1)}{1+2m}\right)^{k^2-m+1}$$

- *For a>0, b<1 solving in the mean-field approximation it can be obtained*

(a=b=1) (a=0,b=1) (a=1,b=0)

$$\langle k^1 \mid k^2\rangle \propto k^2$$



# Class of network models

- Growing networks:
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    Barabasi & Albert 1999,

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- **Static networks:**
  - **Hidden variables mechanism**

    *Bollobas 1979,*

    *Chung & Lu 2002,*

    *Caldarelli et al. 2002,*

    *Park & Newman 2003*

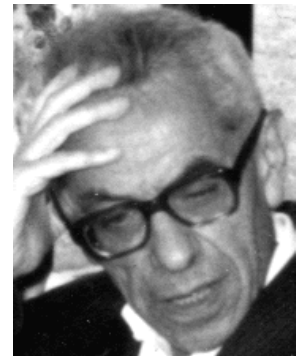
# Random graphs

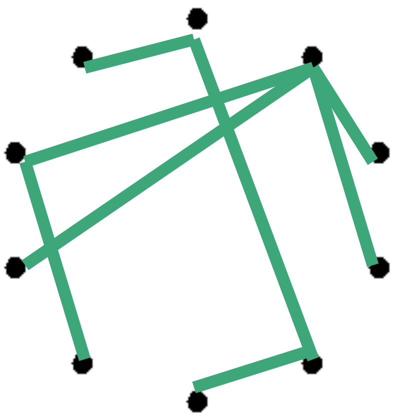
**G(N,L) ensemble**

**Graphs with exactly N nodes and L links**

**G(N,p) ensemble**

**Graphs with N nodes**
**Each pair of nodes linked with probability p**



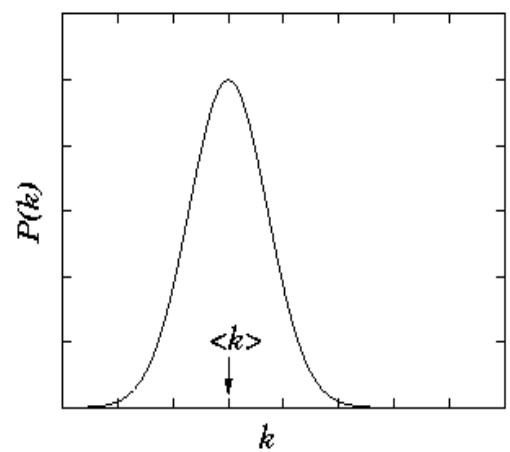


**Binomial distribution**

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

$$P(k) = \frac{1}{k!} c^k e^{-c}$$

**Poisson distribution**



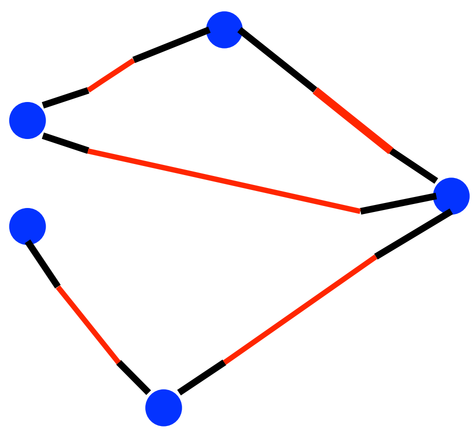
# Statistical mechanics and random graphs

| Statistical mechanics | | Random graphs | |
|---|---|---|---|
| **Microcanonical Ensemble** | **Configurations with fixed energy E** | **G(N,L) Ensemble** | **Graphs with fixed # of links L** |
| **Canonical Ensemble** | **Configurations with fixed average energy <E>** | **G(N,p) Ensemble** | **Graphs with fixed average # of links <L>** |

# Networks with given degree sequence

**Microcanonical ensemble**

$$P(G)=\frac{1}{\Sigma_1}\prod_i \delta(k_i-\sum_j a_{ij})$$

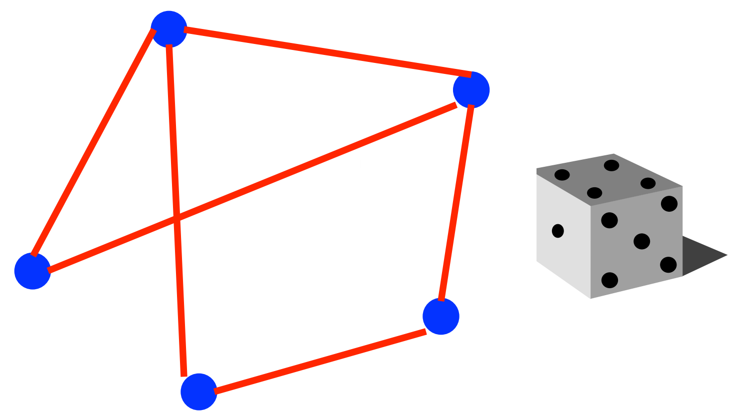


Ensemble of network with exact degree sequence

**Configuration model**

**Canonical ensemble**

$$P(G)=\prod_{i<j} p_{ij}{}^{a_{ij}}(1-p_{ij})^{1-a_{ij}}$$



Ensemble of networks given expected degree sequence

**Hidden variables model**

# Entropy of network ensembles

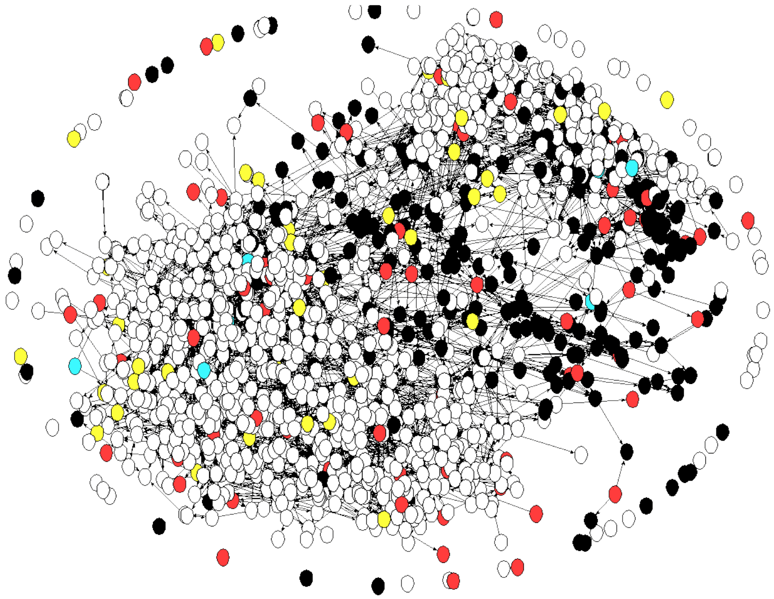
Entropy of a **canonical network ensemble** with linear constraints

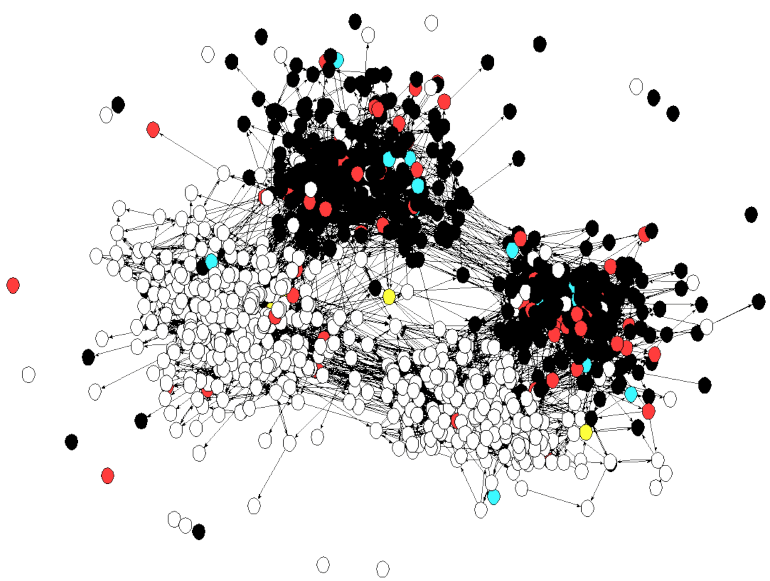
$$S = -\left[ \sum_{ij} p_{ij} \ln p_{ij} + (1 - p_{ij}) \ln (1 - p_{ij}) \right]$$

Entropy of a **microcanonical network ensemble with linear constraints con be found by the cavity method**, in the configuration model for sparse network limit with structural cutoff we recover the Bender and Canfield formula

$$\Sigma = \log[\aleph] \quad with\ \aleph = \frac{(2L)!!}{\prod_i k_i!} e^{-\frac{1}{4}\left(\frac{\langle k^2 \rangle}{\langle k \rangle}\right)^2}$$

**Entropy measures can be used to assess the role of features for network structure**





# $\Theta$
# An independent measure

**Two schools with similar N,M**

**$N_1$=1461 $N_2$=1147**
**$M_1$=.64 $M_2$=.66**

**But different $\Theta$**

$\Theta_1 / N_1^{1/2} = 1.69 \quad \Theta_2 / N_2^{1/2} = 15.71$

**G Bianconi et al. PNAS 2009**

# Multiplex measures: Overlap

- For two layers $\alpha$ and $\alpha'$ of the multiplex we can define the **total overlap $O^{\alpha\alpha'}$** as

$$\boxed{O^{\alpha,\alpha'} = \sum_{i<j} a_{ij}^{\alpha} a_{ij}^{\alpha'}}$$

- For a node i of the multiplex, we can define the **local overlap** $o_i^{\alpha,\alpha'}$

$$\boxed{o_i^{\alpha,\alpha'} = \sum_{j} a_{ij}^{\alpha} a_{ij}^{\alpha'}}$$

# Uncorrelated and correlated multiplex ensembles

- A multiplex $\vec{G}$ can be seen as a set of graphs $G_\alpha$ in each layer a of the multiplex, i.e. $\vec{G} = (G_1, G_2, \ldots G_\alpha, \ldots . G_M)$
- A **uncorrelated multiplex ensemble** assign to every multiplex a probability given by

$$\boxed{P(\vec{G}) = \prod_{\alpha=1,M} P_\alpha(G_\alpha)}$$

- If instead

$$\boxed{P(\vec{G}) \neq \prod_{\alpha=1,M} P_\alpha(G_\alpha)}$$

the **multiplex ensemble is correlated**

# Overlap in uncorrelated mutliplex ensembles

**In every uncorrelated multiplex ensembles formed by sparse networks the global and local overlap are negligible**

G. Bianconi arxiv:1303.4057 (2013)

# Multilinks and Multiadjacency matrices

- Consider a vector $\vec{m} = (m_1, m_2, \ldots m_\alpha, \ldots m_M)\ with \quad m_\alpha = 0,1$
- A **multilink** $\vec{m}$ is the set of links connecting a given pair of nodes in the different layers of the multiplex and connecting them in a generic layer $\alpha$ only if $m_\alpha=1$.
- The **multiadjacency matrices** have elements $A_{ij}^{\vec{m}} = 1$ only if there is a multilink $\vec{m}$ between node i and node j and zero otherwise, i.e.

$$\boxed{A_{ij}^{\vec{m}} = \prod_{\alpha=1,\ldots M} [m_\alpha a_{ij}^\alpha + (1-m_\alpha)(1-a_{ij}^\alpha)]}$$

# Case of two layers

## Multiadjacency matrices

$$A_{ij}^{10} = \begin{cases} 1 & \text{if node } i \text{ and node } j \text{ are linked in layer } 1 \text{ and not linked in layer } 2 \\ 0 & \text{otherwise} \end{cases}$$

$$A_{ij}^{01} = \begin{cases} 1 & \text{if node } i \text{ and node } j \text{ are linked in layer } 2 \text{ and not linked in layer } 1 \\ 0 & \text{otherwise} \end{cases}$$

$$A_{ij}^{11} = \begin{cases} 1 & \text{if node } i \text{ and node } j \text{ are linked in layer } 1 \text{ and in layer } 2 \\ 0 & \text{otherwise} \end{cases}$$

$$A_{ij}^{00} = \begin{cases} 1 & \text{if node } i \text{ and node } j \text{ are not linked in layer } 1 \text{ and not linked in layer } 2 \\ 0 & \text{otherwise} \end{cases}$$

## Constraints on the multiadjacency matrices

$$\boxed{A_{ij}^{10} + A_{ij}^{01} + A_{ij}^{11} + A_{ij}^{00} = 1}$$

# Multidegree

- The multidegree $\vec{m}$ is defined as

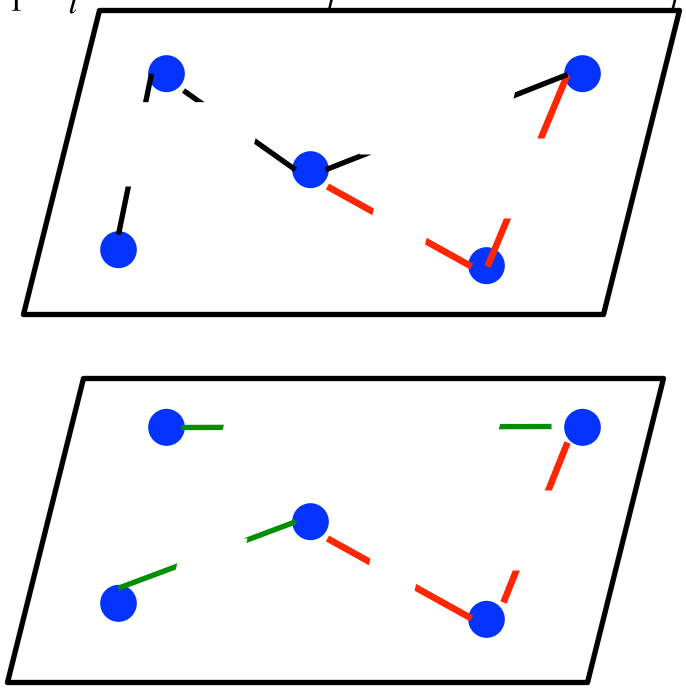
$$k_i^{\vec{m}} = \sum_j A_{ij}^{\vec{m}}$$

- In the case of two layers we have

$$k_i^{10} = \sum_j a_{ij}^1 (1 - a_{ij}^2)$$
$$k_i^{01} = \sum_j (1 - a_{ij}^1) a_{ij}^2$$
$$k_i^{11} = \sum_j a_{ij}^1 a_{ij}^2 = o_i$$

# Configuration model for the correlated multiplex(microcanonical ensemble)

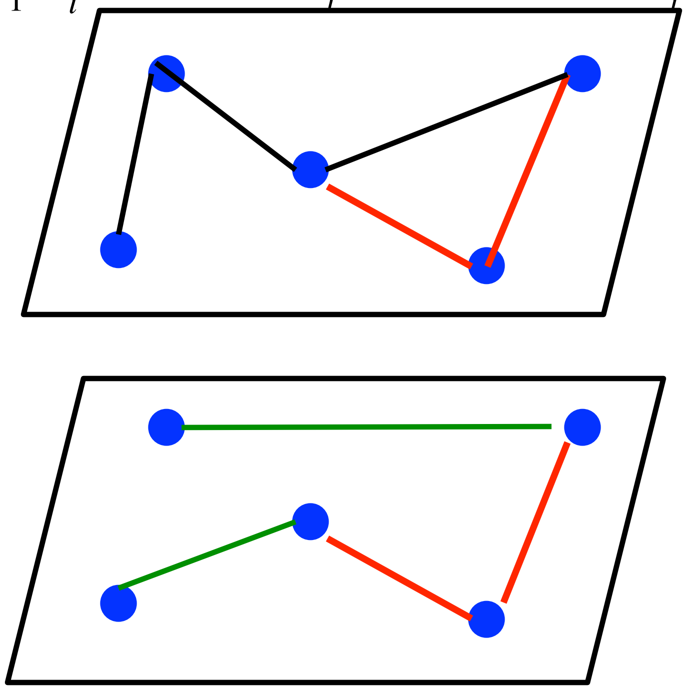
$$P(\vec{G}) = \frac{1}{\Sigma_1}\prod_i \delta\left(k^{10}{}_i - \sum_j A^{10}_{ij}\right)\delta\left(k^{01}{}_i - \sum_j A^{01}_{ij}\right)\delta\left(k^{11}{}_i - \sum_j A^{11}_{ij}\right)$$



Ensemble of multiplex with given multidegree sequence

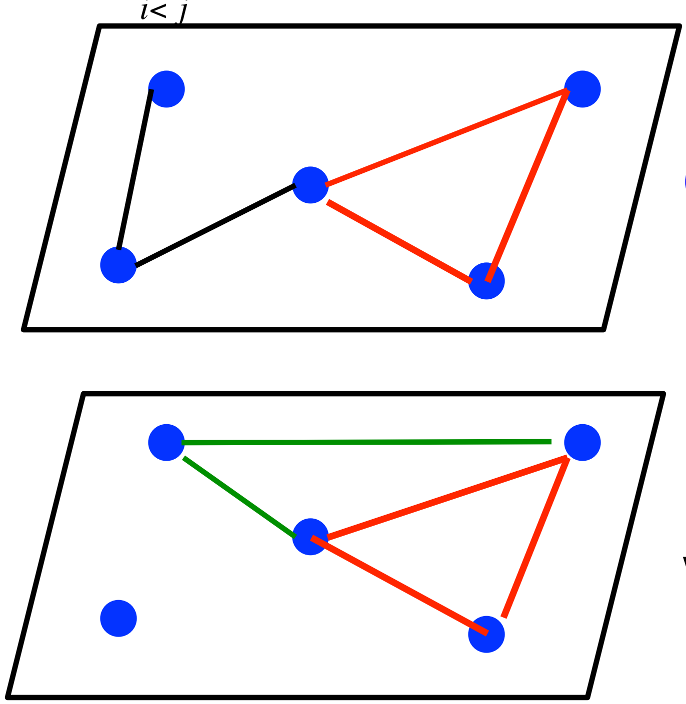
# Configuration model for the correlated multiplex (microcanonical ensemble)

$$P(\vec{G}) = \frac{1}{\Sigma_1}\prod_i \delta(k^{10}{}_i - \sum_j A_{ij}^{10})\delta(k^{01}{}_i - \sum_j A_{ij}^{01})\delta(k^{11}{}_i - \sum_j A_{ij}^{11})$$



# Canonical network model for the correlated multiplex

$$P(\vec{G}) = \prod_{i<j} (p_{ij}^{10} A_{ij}^{10} + p_{ij}^{01} A_{ij}^{01} + p_{ij}^{11} A_{ij}^{11} + p_{ij}^{00} A_{ij}^{00})$$



**Constructive algorithm**

**For every pair of nodes (i,j)**

**Draw a multilink** $\vec{m}$

**with probability** $p_{ij}^{\vec{m}}$ **,**

**i.e. put a link in every layer**

**where** $m_\alpha$**=1.**

**G. Bianconi arxiv:1303.4057 (2013)**

# Entropy of correlated multiplex ensembles

Entropy of a **canonical multiplex ensemble** with linear constraints

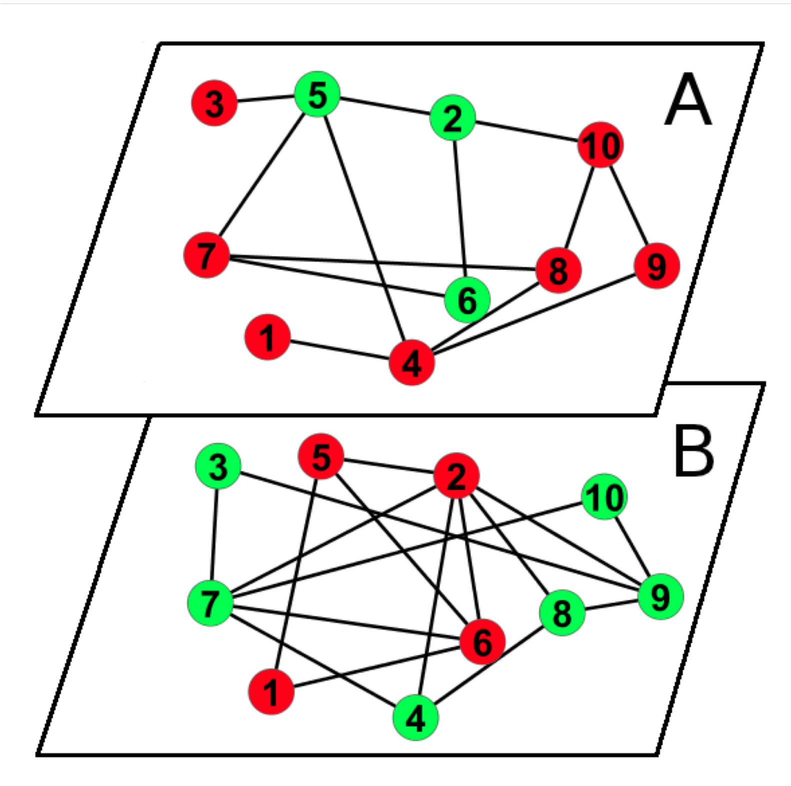
$$S = -\left[\sum_{\vec{m}} \sum_{ij} p_{ij}^{\vec{m}} \ln p_{ij}^{\vec{m}}\right]$$

Entropy of a **microcanonical mutliplex ensemble with linear constraints con be found by the cavity method**, if we fix only the multi degree sequence in the sparse network limit, with structural cutoff we recover a generalization of the Bender and Canfield formula

$$\Sigma = \log[\aleph] \quad with\ \aleph = \prod_{\vec{m} \mid \sum_{\alpha} m_{\alpha} > 0} \frac{(2L^{\vec{m}})!!}{\prod_i k_i^{\vec{m}}!} e^{-\frac{1}{4}\left(\frac{\left\langle (k^{\vec{m}})^2 \right\rangle}{\left\langle k^{\vec{m}} \right\rangle}\right)^2}$$

# Election model on multiplex networks

# Competing networks: the case of political elections

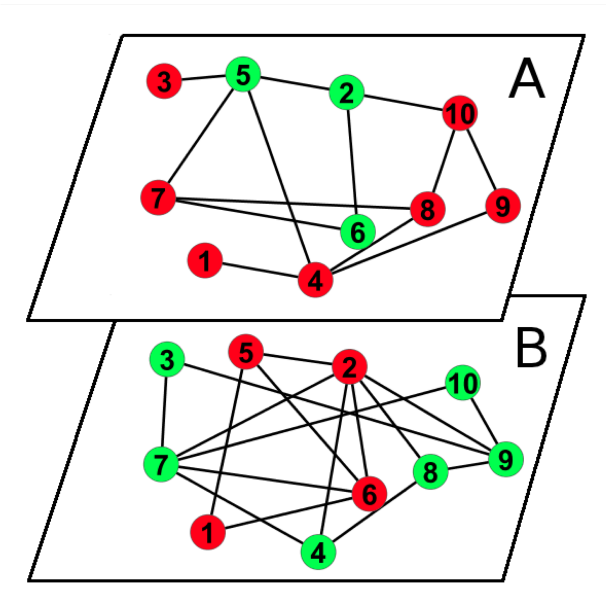



We assume that each party is represented by a network.

At the end of the election campaign each agent can be active at most in one network (he/she will vote for the corresponding party)

The election campaign is described as a competition between the two networks.

# The election model




- The two layers are competing – on the election day, agents must be either active (green) in only one of the networks, or inactive (red) in both.
- At the election day an agent is voting for one party if at least one neighbor is voting for that party

$$\sigma_i^A = \left[1 - \prod_{j \in N_A(i)} (1 - \sigma_j^A)\right](1 - \sigma_i^B)$$

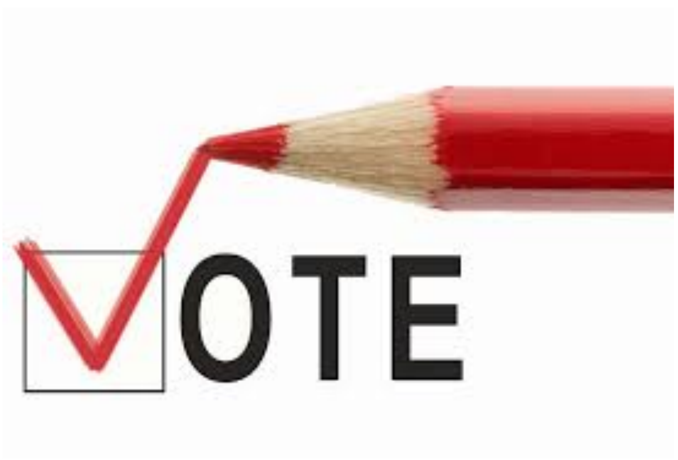
$$\sigma_i^B = \left[1 - \prod_{j \in N_B(i)} (1 - \sigma_j^B)\right](1 - \sigma_i^A)$$

- During the "campaigning" process, we count the number of voters that conflict with the criterion above with the Hamiltonian

$$\boxed{H = \sum_i \left\{ \sigma_i^A - \left[1 - \prod_{j \in N_A(i)} (1 - \sigma_j^A)\right](1 - \sigma_i^B) \right\}^2 + \sum_i \left\{ \sigma_i^B - \left[1 - \prod_{j \in N_B(i)} (1 - \sigma_j^B)\right](1 - \sigma_i^A) \right\}^2}$$

# The election model

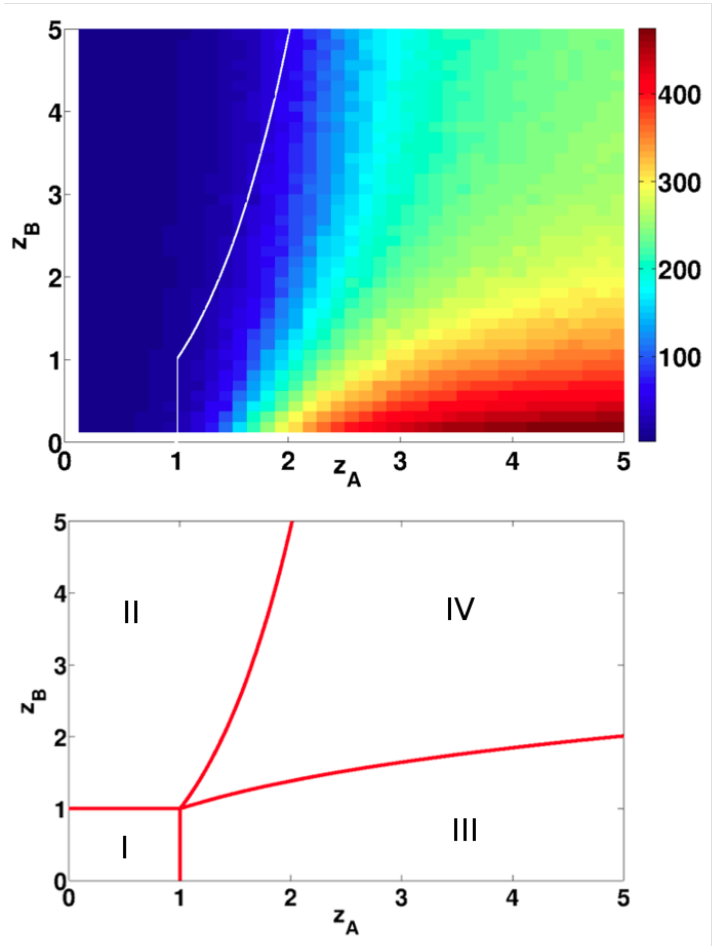
- At the beginning of the campaigning process, voters are "undecided" with random opinions subject to change.
- Evolution of opinions is modeled by a simulated annealing process.




As the election day approaches the temperature of the simulated annealed algorithm is reduced.

H has multiple fundamental states, simulated annealing always converges to one of these at the election day.

# Phase diagram of the model



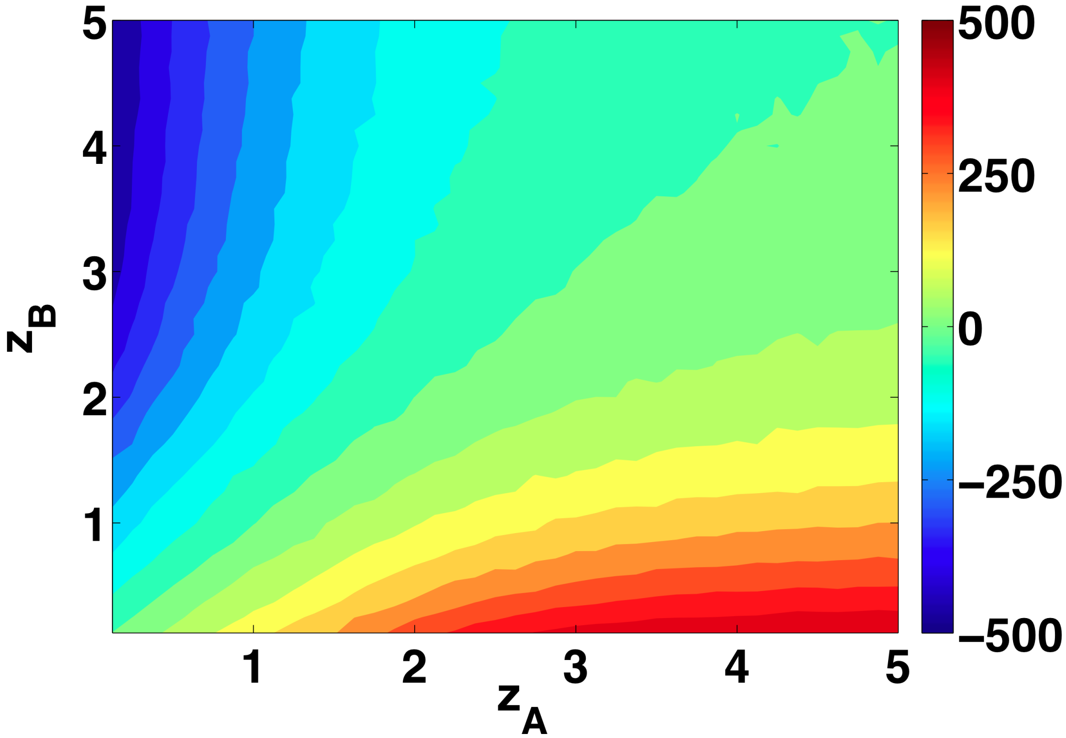

Two E-R networks are considered with average connectivities $z_A$ and $z_B$.
The size of the giant component of the percolating cluster in network A is plotted as a function of the average connectivities.

Region(I): $S_A=0$, $S_B=0$
Region(II): $S_A=0$ $S_B>0$
Region(III): $S_A>0$ $S_B=0$
Region(IV): $S_A>0$ $S_B>0$

**In region III both political parties percolate in the population**

**Halu, Zhao, Baronchelli and Bianconi EPL (2013)**

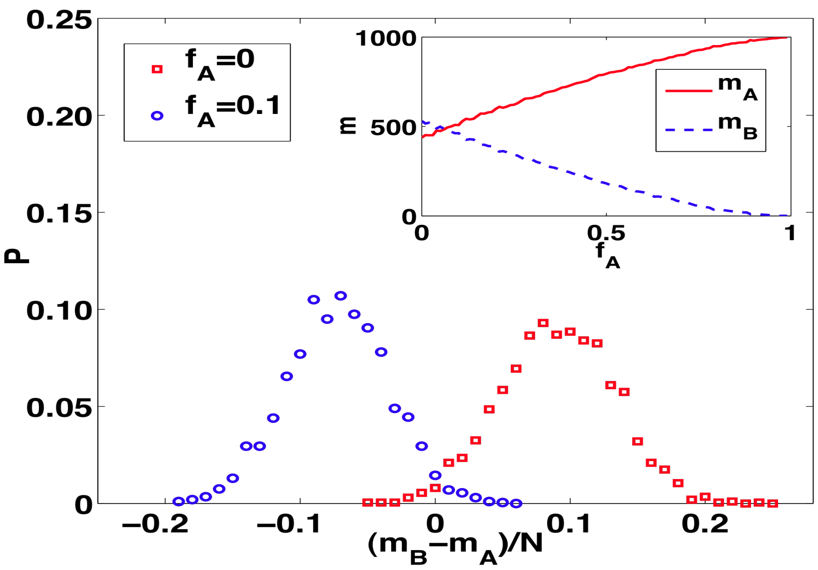
# Connect and win




We plot $m_A$-$m_B$, i.e. the number of agents that vote for party A minus the number of agents that vote for party B averaged over 90 realizations.

The most connected party is the one that is more likely to win the election!

# Effect of the committed agents in the majority




- The effect of committed agents in Region II ( $z_A$=2.5 $z_B$=4)
- A small fraction (~0.1) of agents is sufficient to reverse the outcome of the election.

Halu, Zhao, Baronchelli, Bianconi EPL (2013)

# Conclusions

- Many **networks  interact, coexist and coevolve with other networks forming multiplexes** where any pair of nodes can be linked by different types of interaction

- **Modeling interacting and multiplex networks** is only in its infancy and we need  to develop a new series of non-equilibrium and equilibrium models and to compare their outcome to real data.

- **Critical phenomena** on multiplex and interacting networks show new surprising physics.

# References and collaborators

- **Growing multiplex model**

**V. Nicosia, G. Bianconi, V. Latora, M. Barthelemy arxiv:1302.7126**

**V. Nicosia (QMUL)**

**V. Latora (QMUL)**

**M. Barthelemy (CEA)**

- **Multiplex ensembles**

**G. Bianconi arxiv:1303.4057**

- **Election model**

**A. Halu, K. Zhao, A. Barochelli and G. Bianconi EPL 102, 16002 (2013)**

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